Homotopy Groups of Spheres

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Outline

- Smooth Structures on Spheres

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- Stable Homotopy Groups of Spheres

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- Motivic Homotopy Theory

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- Motivic Homotopy Theory
- Future Directions

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- $n \geqslant 5$, Smale (smooth), Newman, Connell. 1960's.

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1. For which n, is there a unique smooth structure on S^n?
2. How many smooth structures are there on S^n?

Kervaire–Milnor $n \geqslant 5$

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Theorem (Kervaire–Milnor)

For $n \geqslant 5$, the subgroup Θ_n^{bp} is cyclic,

$$|\Theta_n^{bp}| = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 1 \text{ or } 2, & \text{if } n = 4k+1, \\ b_k, & \text{if } n = 4k-1. \end{cases}$$

$b_k = 2^{2k-2}(2^{2k-1}-1)\cdot$ *the numerator of* $\frac{4B_{2k}}{k}$,
B_{2k}: *Bernoulli number.*

Theorem (Kervaire–Milnor)

(continued) Suppose $n \geqslant 5$*.*

1. *For* $n \not\equiv 2 \pmod 4$*, there is an exact sequence*

$$0 \to \Theta_n^{bp} \to \Theta_n \to \pi_n/J \to 0.$$

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- The Kervaire Invariant Problem: For which n, $\Phi_n \neq 0$?

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- The Arf invariant classifies isomorphic classes of non-singular quadratic forms over $\mathbb{Z}/2$.

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$$\text{framed manifolds of dim } n \longleftrightarrow \pi_n$$
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- (Lin–Wang–Xu 2024): There exists M with $\Phi(M) = 1$ in dim 126.

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⇒ Odd dimensional spheres that *could* have a unique smooth structure:

$$S^1, S^3, S^5, S^{13}, S^{29}, S^{61}, S^{125}$$

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Based on work of Kervaire–Milnor, Browder, Hill–Hopkins–Ravenel,

Corollary

The only odd dimensional spheres with a unique smooth structure are S^1, S^3, S^5, S^{61}.

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  - Towards 100%: ongoing progress with Behrens, et al.

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- (Serre)

$$\pi_{n+k}(S^k) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \textit{if} \quad n = 0 \\ \mathbb{Q}, & \textit{if} \quad k \textit{ is even and } n = k-1, \\ 0, & \textit{else}. \end{cases}$$

Low dimensional computations

$\pi_{0+n}(S^n)$	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$\pi_{1+n}(S^n)$	·	·	∞	2	2	2	2	2	2	2	2	2	2	2
$\pi_{2+n}(S^n)$	·	·	2	2	2	2	2	2	2	2	2	2	2	2
$\pi_{3+n}(S^n)$	·	·	2	12	$\infty \cdot 12$	24	24	24	24	24	24	24	24	24
$\pi_{4+n}(S^n)$	·	·	12	2	2^2	2	·	·	·	·	·	·	·	·
$\pi_{5+n}(S^n)$	·	·	2	2	2^2	2	∞	·	·	·	·	·	·	·
$\pi_{6+n}(S^n)$	·	·	2	3	$24 \cdot 3$	2	2	2	2	2	2	2	2	2
$\pi_{7+n}(S^n)$	·	·	3	15	15	30	60	120	$\infty \cdot 120$	240	240	240	240	240
$\pi_{8+n}(S^n)$	·	·	15	2	2	2	$24 \cdot 2$	2^3	2^4	2^3	2^2	2^2	2^2	2^2
$\pi_{9+n}(S^n)$	·	·	2	2^2	2^3	2^3	2^3	2^4	2^5	2^4	$\infty \cdot 2^3$	2^3	2^3	2^3
$\pi_{10+n}(S^n)$	·	·	2^2	$12 \cdot 2$	$120 \cdot 12 \cdot 2$	$72 \cdot 2$	$72 \cdot 2$	$24 \cdot 2$	$24^2 \cdot 2$	$24 \cdot 2$	$12 \cdot 2$	$6 \cdot 2$	6	6
$\pi_{11+n}(S^n)$	·	·	$12 \cdot 2$	$84 \cdot 2^2$	$84 \cdot 2^5$	$504 \cdot 2^2$	$504 \cdot 4$	$504 \cdot 2$	$504 \cdot 2$	$504 \cdot 2$	504	504	$\infty \cdot 504$	504

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- (Serre) π_n are finite abelian groups for $n > 0$.
  $\Rightarrow$ Compute one prime at a time.
- $\pi_* = \bigoplus_n \pi_n$ has a richer structure!
  - π_* form a ring by composition.

$$\pi_l \otimes \pi_m \longrightarrow \pi_{l+m}$$

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Theorem (Freudenthal)

When $k \geqslant n+2$, the groups $\pi_{n+k}(S^k)$ only depend on n, not k.

We call this group the n-th stable homotopy groups of spheres, or the n-th stem, denoted by π_n.

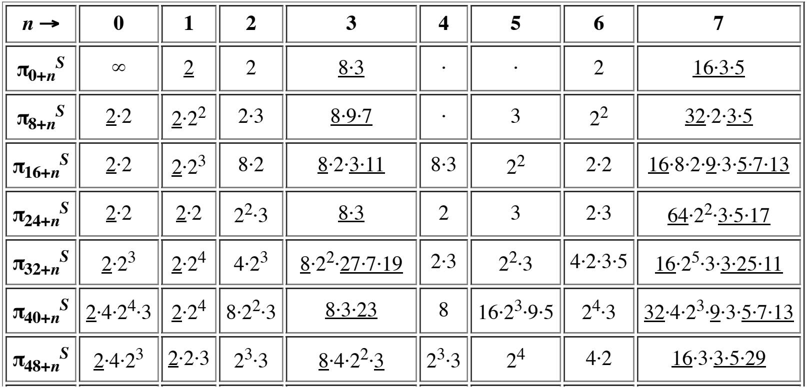
- (Serre) π_n are finite abelian groups for $n > 0$.
  $\Rightarrow$ Compute one prime at a time.
- $\pi_* = \bigoplus_n \pi_n$ has a richer structure!
  - π_* form a ring by composition.

$$\pi_l \otimes \pi_m \longrightarrow \pi_{l+m}$$

  - Higher products: (matric) Toda brackets

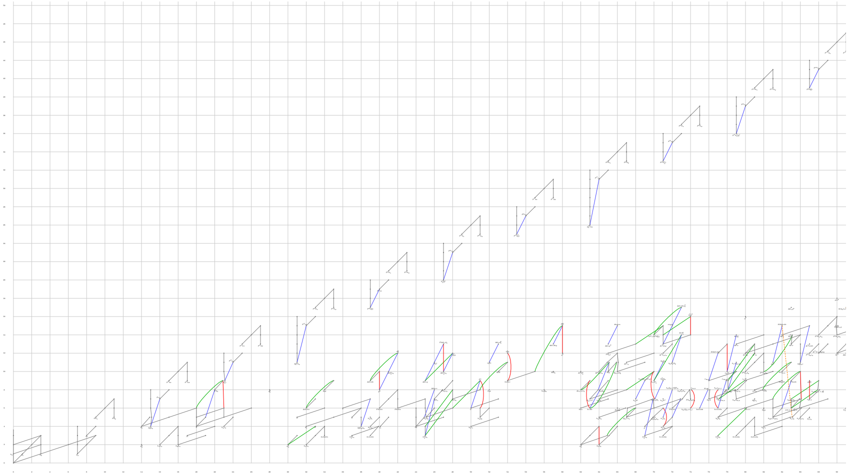
$$\pi_l \otimes \pi_m \otimes \pi_n \longrightarrow \pi_{l+m+n-1}$$

Stable stems

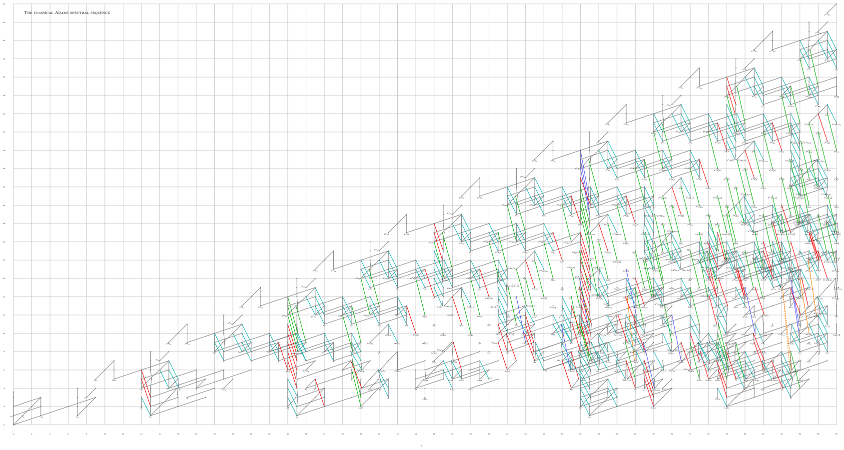


$n \rightarrow$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
π_{0+n}^S	∞	$\underline{\underline{2}}$	2	$\underline{8\cdot 3}$	$\cdot$	$\cdot$	2	$\underline{16\cdot 3\cdot 5}$
π_{8+n}^S	$\underline{\underline{2}}\cdot 2$	$\underline{\underline{2}}\cdot 2^2$	$2\cdot 3$	$\underline{8\cdot 9\cdot 7}$	$\cdot$	3	2^2	$\underline{32}\cdot 2\cdot \underline{3\cdot 5}$
π_{16+n}^S	$\underline{\underline{2}}\cdot 2$	$\underline{\underline{2}}\cdot 2^3$	$8\cdot 2$	$\underline{8}\cdot 2\cdot \underline{3\cdot 11}$	$8\cdot 3$	2^2	$2\cdot 2$	$\underline{16}\cdot 8\cdot 2\cdot \underline{9}\cdot 3\cdot \underline{5\cdot 7\cdot 13}$
π_{24+n}^S	$\underline{\underline{2}}\cdot 2$	$\underline{\underline{2}}\cdot 2$	$2^2\cdot 3$	$\underline{8\cdot 3}$	2	3	$2\cdot 3$	$\underline{64}\cdot 2^2\cdot \underline{3\cdot 5\cdot 17}$
π_{32+n}^S	$\underline{\underline{2}}\cdot 2^3$	$\underline{\underline{2}}\cdot 2^4$	$4\cdot 2^3$	$\underline{8}\cdot 2^2\cdot \underline{27\cdot 7\cdot 19}$	$2\cdot 3$	$2^2\cdot 3$	$4\cdot 2\cdot 3\cdot 5$	$\underline{16}\cdot 2^5\cdot 3\cdot \underline{3\cdot 25\cdot 11}$
π_{40+n}^S	$\underline{\underline{2}}\cdot 4\cdot 2^4\cdot 3$	$\underline{\underline{2}}\cdot 2^4$	$8\cdot 2^2\cdot 3$	$\underline{8\cdot 3\cdot 23}$	8	$16\cdot 2^3\cdot 9\cdot 5$	$2^4\cdot 3$	$\underline{32}\cdot 4\cdot 2^3\cdot \underline{9}\cdot 3\cdot \underline{5\cdot 7\cdot 13}$
π_{48+n}^S	$\underline{\underline{2}}\cdot 4\cdot 2^3$	$\underline{\underline{2}}\cdot 2\cdot 3$	$2^3\cdot 3$	$\underline{8}\cdot 4\cdot 2^2\cdot \underline{3}$	$2^3\cdot 3$	2^4	$4\cdot 2$	$\underline{16}\cdot 3\cdot \underline{3\cdot 5\cdot 29}$

2-primary computations



The Adams Spectral Sequence



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$$E_2^{s,t} = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s}(S^0)_p^\wedge$$

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  MU: complex cobordism spectrum

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  Any method that computes nontrivial differentials in such a spectral sequence will leave infinitely many differentials undecided.

$$\begin{tikzcd}
 & \pi_*(S^0)^\wedge_p & \\
\mathrm{Ext}^{*,*}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[ur, Rightarrow, "\textbf{Adams SS}" description] & & \mathrm{Ext}^{*,*}_{\mathrm{MU}_*\mathrm{MU}}(\mathrm{MU}_*, \mathrm{MU}_*)^\wedge_p \arrow[ul, Rightarrow, "\textbf{Adams--Novikov SS}" description]
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Theorem (Miller)

Adams d_2 differentials $\longleftrightarrow$ algebraic Novikov d_2 differentials

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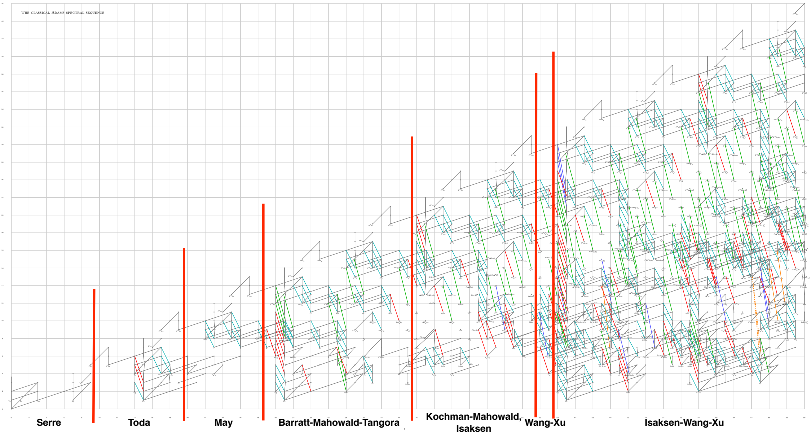
- (2023 - now)
    - (Lin–Wang–Xu)
      ongoing progress towards the last Kervaire invariant problem in dimension 126 and beyond

Classical Adams Spectral Sequence up to 90-stem



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- Motivic analogue of classical computational tools exist!

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Isaksen's computation
up to 60-stem

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The same data!

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\mathrm{Ext}^{s,2w}_{P_*}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) \arrow[r, "\cong"] \arrow[u, Rightarrow, "\textbf{algebraic Novikov SS}" description] & \mathrm{Ext}^{a,2w-s+a,w}_{A^{mot}_{*,*}}(\mathbb{F}_p, \mathbb{F}_p[\tau]) \arrow[u, Rightarrow, "\textbf{motivic Adams SS}" description]
\end{tikzcd}$$

Theorem (Gheorghe–Wang–Xu)

The above two spectral sequences are isomorphic.

τ as a deformation parameter

Theorem (Gheorghe–Wang–Xu)

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 & \mathrm{Ext}^{*,*}_{P_*}(\mathbb{F}_p, I^*/I^{*+1}) \arrow[ul, Rightarrow, "\textbf{Cartan-Eilenberg SS}" description] \arrow[ur, Rightarrow, "\textbf{algebraic Novikov SS}" description] &
\end{tikzcd}$$

Theorem (Miller)

Adams d_2 differentials $\longleftrightarrow$ algebraic Novikov d_2 differentials

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{Adams SS}"] & \mathrm{Ext}^{*,*,*}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p[\tau]) \arrow[l] \arrow[r] \arrow[d, Rightarrow, "\textbf{motAdamsSS}"] & \mathrm{Ext}^{*,*,*}_{A^{mot}_{*,*}}(\mathbb{F}_p[\tau], \mathbb{F}_p) \arrow[d, Rightarrow, "\textbf{motAdamsSS}"] \\
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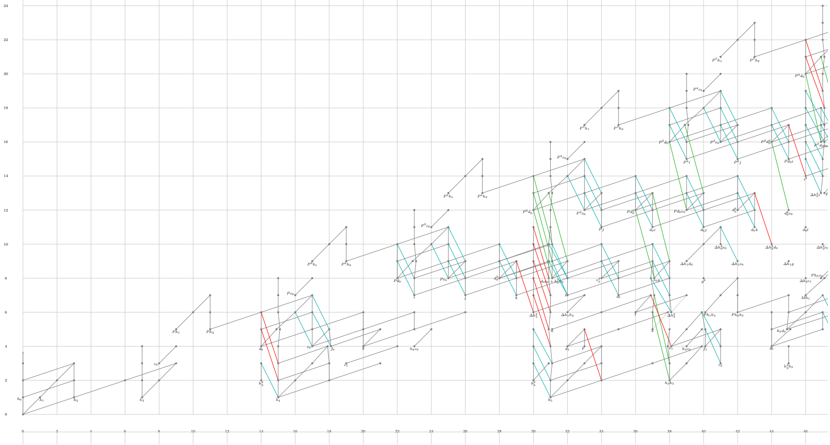
Re-compute early range very effectively

Classical Adams spectral sequence



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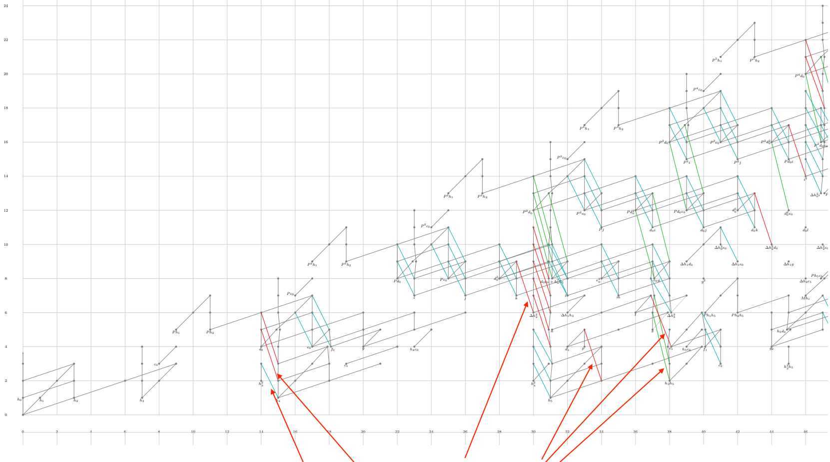
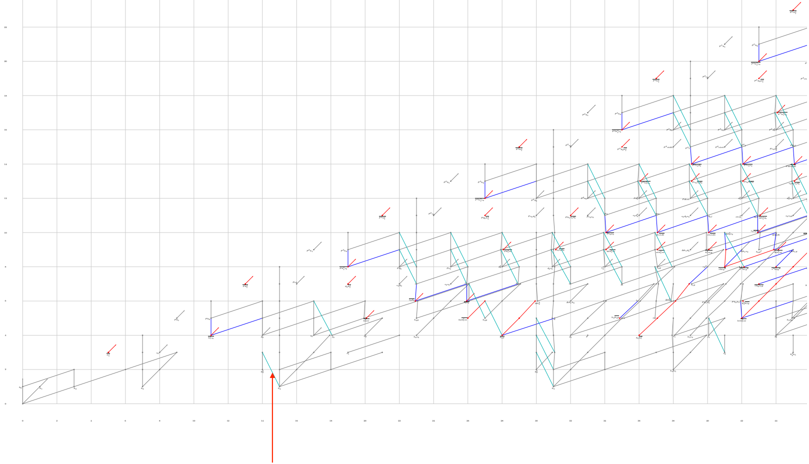
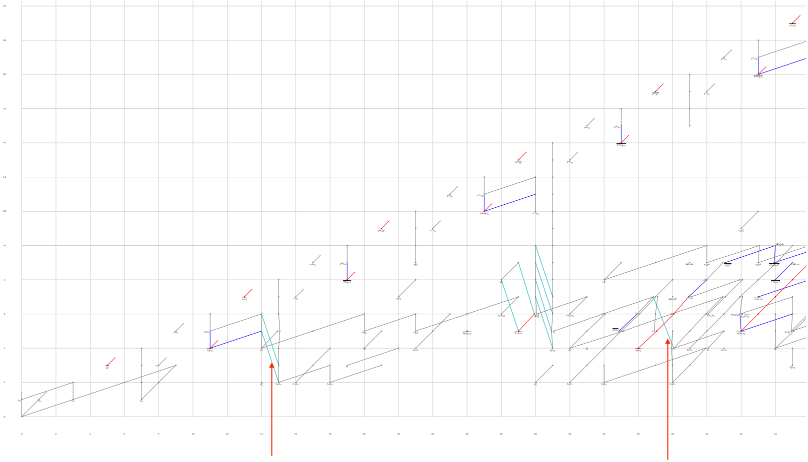


Harder Differentials

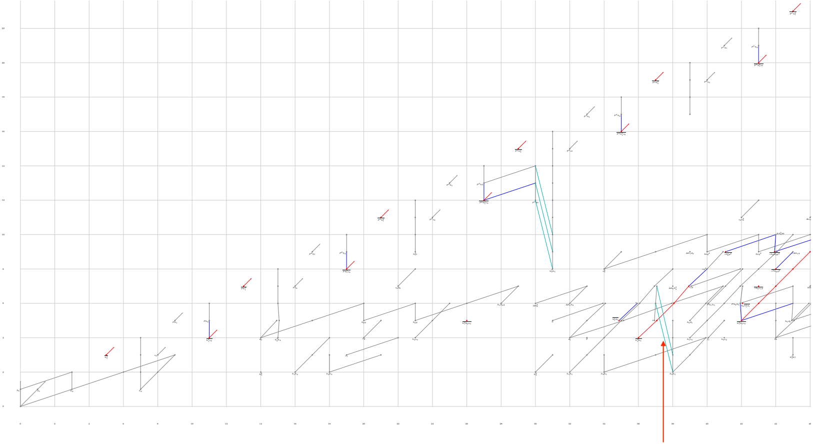
Motivic E_2-page of $\widehat{S^{0,0}}/\tau$



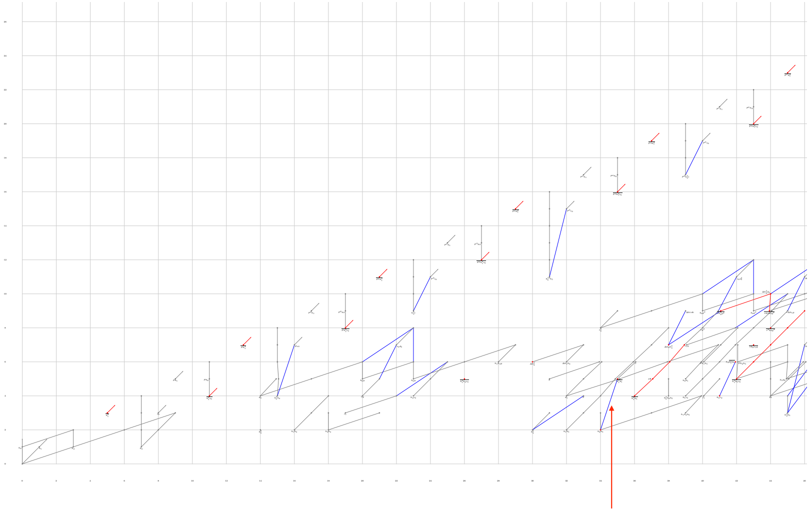
Motivic E_3-page of $\widehat{S^{0,0}}/\tau$



Motivic E_4-page of $\widehat{S^{0,0}}/\tau$



Motivic E_∞-page of $\widehat{S^{0,0}}/\tau$



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So it does not violate the Second Mahowald Uncertainty Principle!

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Theorem (Bachmann–Kong–Wang–Xu)

Let $E \in \mathsf{SH}(k)$.

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Postnikov–Whitehead Tower

Postnikov–Whitehead tower for $S^{0,0}$ w.r.t. the Chow t-structure:

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 & {} \arrow[d] & \\
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Computing $\pi_{*,*}S^{0,0}$ over k

Apply the motivic Adams spectral sequences:

$$\begin{tikzcd}
 & {} \arrow[d] & & \\
 & \mathbf{motASS}(S^{0,0}_{c\geqslant 2}) \arrow[r] \arrow[d] & \mathbf{motASS}(S^{0,0}_{c=2}) = \mathbf{algNSS}((\mathrm{MGL}_{*,*})_{c=2}) & \\
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\mathbf{motASS}(S^{0,0}) \arrow[r, phantom, "="] & \mathbf{motASS}(S^{0,0}) \arrow[r] & \mathbf{motASS}(S^{0,0}_{c=0}) \arrow[r, equal] & \mathbf{algNSS}(\mathrm{MU}_*)
\end{tikzcd}$$

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- Ongoing progress: interpretation in terms of framed manifolds.

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- Uniform Doomsday Conjecture: For any nonzero Sq^0-family $\{a_j\}$, there exists a Sq^0-family $\{b_j\}$, $r \geqslant 2$, $c \in \mathrm{Ext}$, such that

$$d_r(a_j) = c \cdot b_j \neq 0,\ \text{for } j >> 0.$$

Thank you!